Surface differential operator and electromagnetic field tensor

1976 J. Phys. A: Math. Gen. 9 301

(http://iopscience.iop.org/0305-4470/9/2/014)

J. Phys. A: Math. Gen., Vol. 9, No. 2, 1976. Printed in Great Britain. © 1976

Surface differential operator and electromagnetic field tensor

A K Pandey and A N Mantri
Nuclear Physics Section, Department of Physics, Banaras Hindu University, Varanasi-221005, India

Received 8 July 1975, in final form 15 October 1975

Abstract. In view of the recent work of Ross concerning Mandelstam's formulation of gauge-independent quantum electrodynamics, and involving the additional assumption of path independence, we suggest the use of a surface differential operator for the electromagnetic field quantities and discuss some of the results so obtained.

Recently, Ross (1974) has shown that if in the Mandelstam (1962) formulation of gauge-independent quantum electrodynamics an assumption of path independence is also made, the electric charge must be quantized. In this paper we show that it is possible to introduce an operator formalism for electromagnetic field quantities thus enabling an alternative interpretation to be given to the work of Ross. We first briefly recall the contents of Ross's paper.

The gauge-invariant field variable at field point x for a spinless particle with charge q is given by

$$\Phi(x, \mathrm{P}) = \phi(x) \exp\left(-\frac{\mathrm{i}q}{\hbar}\int_{\mathrm{P}}^{x} \mathrm{d}\xi_\mu A_\mu(\xi)\right) \tag{1}$$

where the integral is taken over the space-like path P from $-\infty$ to the point x, ϕ satisfies the usual Klein–Gordon equation, and A_μ is the vector potential. Considering two paths P_1 and P_2 to the field point x from $-\infty$ which are the same everywhere except between the two points where they separate, Ross has shown that the requirement of path independence leads to the condition

$$\exp\left(-\frac{\mathrm{i}q}{\hbar}\oint_{\mathrm{C}} \mathrm{d}\xi_\mu A_\mu(\xi)\right) = 1 \tag{2}$$

i.e.

$$q\oint_{\mathrm{C}} \mathrm{d}\xi_\mu A_\mu(\xi) = 2\pi n\hbar \tag{2a}$$

where C is the closed path P_2–P_1 and n is an integer. Equation (2a) then leads to the quantization of charge.

Instead of working with a potential A_μ one can introduce, following Cabibbo and Ferrari (1962), the electromagnetic field tensor $F_{\mu\nu}$ in the theory and one gets

$$\Phi(x, \mathrm{P}_2) = \Phi(x, \mathrm{P}_1) \exp\left(-\frac{\mathrm{i}q}{2\hbar}\int_{\mathrm{S}} \mathrm{d}\sigma_{\mu\nu}F_{\mu\nu}\right) \tag{3}$$

where S is the surface delimited by the closed path C. The path independence then gives

$$\exp\left(-\frac{iq}{2\hbar}\int_S d\sigma_{\mu\nu}F_{\mu\nu}\right)=1 \tag{4}$$

i.e.

$$\frac{q}{2}\int_S d\sigma_{\mu\nu}F_{\mu\nu}=2\pi n\hbar \tag{4a}$$

which can be obtained from (2), (2a) by using a relativistic generalization of Stokes' theorem. Ross has suggested that equation (2a) can be looked upon as an operator eigenvalue equation. The similarity of equations (2a) and (4a) to the usual Bohr quantization conditions also indicates that it may be possible to obtain a representation of the field quantities in terms of operators and our findings in this direction are given below.

Consider equation (3). We work in the Cartesian system. The surface S can be characterized by its six components $S_{\mu\nu}$, which are the projections of the surface on six planes $x_\mu x_\nu$, $\mu\neq\nu$. One can write equation (3) as

$$\Phi(x,P_2)=\Phi(x,P_1)\psi(S),$$

or explicitly as

$$\Phi(x,P_2)=\Phi(x,P_1)\psi(S_{\mu\nu}). \tag{5}$$

It is easy to see that $\psi(S_{\mu\nu})$ satisfies the following surface differential equation:

$$\left[\frac{\partial}{\partial S_{\mu\nu}}\psi(S_{\mu\nu})\right]_P=-\frac{iq}{\hbar}F_{\mu\nu}(P)\psi(S_{\mu\nu}) \tag{6}$$

where the surface derivatives are defined (Killingbeck and Cole 1972) analogous to the usual definition of the derivative, i.e.

$$\left[\frac{\partial\psi}{\partial S_{\mu\nu}}\right]_P=\lim_{\delta S_{\mu\nu;P}\to 0}\left(\frac{\psi(S_{\mu\nu}+\delta S_{\mu\nu;P})-\psi(S_{\mu\nu})}{\delta S_{\mu\nu;P}}\right) \tag{7}$$

where variation of the surface is made in the neighbourhood of the boundary point P.

Equation (6) suggests that it may be possible to make the following operator replacement for the electromagnetic field tensor $F_{\mu\nu}$:

$$F_{\mu\nu}\to\frac{i\hbar}{q}\frac{\partial}{\partial S_{\mu\nu}}. \tag{8}$$

At this stage one is reminded of the operator replacement of observables in quantum mechanics and corresponding uncertainty relations. It may not be a coincidence that there also exist uncertainty relations (Furry and Ramesey 1960, Weisskopf 1961) concerning measurements of electric and magnetic fields. Thus for example Heitler (1954) has shown that the inaccuracy ΔE_x in the electric field strength measurement is given by

$$\Delta E_x=\hbar/\epsilon\,\Delta xT \tag{9}$$

where ϵ is the charge of the test body, Δx is the unknown displacement of the test body and T is the time interval for the measurement. It is easy to see that equation (9) is the component of the general relation

$$\Delta F_{\mu\nu}\,\Delta S_{\mu\nu} \geqslant \hbar/q \tag{10}$$

a consequence of the operator replacement (8). A similar relation involving the magnetic field strength can also be derived, giving support to our suggestion.

Next we come to the path-independence concept of Ross leading to charge quantization. In our operator formalism it can be interpreted as a boundary condition on the function $\psi(S)$. Also the physical example of quantization of flux in superconductors discussed by Ross in support of his path-independence argument is nothing but the simple problem of a particle in a box, as the non-penetration of the magnetic field is equivalent to having an infinitely high potential wall and the confinement of the particle then leads to flux quantization.

Acknowledgments

One of us (AKP) would like to thank Professors B M Udgaonkar, J V Narlikar and G Rajshekharan of the Tata Institute of Fundamental Research for useful discussions, and the Council of Scientific and Industrial Research for financial support in the form of a research fellowship.

References

Cabibbo N and Ferrari E 1962 *Nuovo Cim.* **23** 1147–51
Furry W H and Ramesey N F 1960 *Phys. Rev.* **118** 623–6
Heitler W 1954 *The Quantum Theory of Radiation* (Oxford: Clarendon) pp 83–4
Killingbeck J and Cole G H A 1972 *Mathematical Techniques and Physical Applications* (Academic: New York) pp 252–3
Mandelstam S 1962 *Ann. Phys., NY* **19** 1–24
Ross D K 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 705–9
Weisskopf V F 1961 *Lectures in Theoretical Physics* (New York: Interscience) p 63